Exercise 4.21 Gaussian decision boundaries

(Source: (Duda et al. 2001), Q3.7).)

Let $p(x \mid y=j) = N\left(\mu_j, \sigma_j\right)$ where $j = 1, 2$ and $\mu_1 = 0, \sigma_1^{\,2} = 1, \mu_2 = 1, \sigma_2^{\,2} = 10^6$.

Let the class priors be equal, $p\left(y=1\right) = p\left(y=2\right) = 0.5$.

a. Find the decision region $R_1 = \left\{x : p\left(x \mid \mu_1, \sigma_1\right) \geq p\left(x \mid \mu_2, \sigma_2\right)\right\}$.
   Sketch the result. Hint: draw the curves and find where they intersect. Find both solutions of the equation $p\left(x \mid \mu_1, \sigma_1\right) = p\left(x \mid \mu_2, \sigma_2\right)$

   Hint: recall that to solve a quadratic equation $a\,x^2 + b\,x + c = 0$, we use $x = \dfrac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$

$$\frac{1}{\sqrt{2\pi\sigma_1^{\,2}}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right) = \frac{1}{\sqrt{2\pi\sigma_2^{\,2}}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)$$

$$\frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^{\,2}}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_2^{\,2}}}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)$$

$$\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right) = \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)$$

$$\frac{1}{\sigma_1}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right) = \frac{1}{\sigma_2}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)$$

$$\log\left(\frac{1}{\sigma_1}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right)\right) = \log\left(\frac{1}{\sigma_2}\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)\right)$$

$$\log\left(\frac{1}{\sigma_1}\right) + \log\left(\exp\left(-\frac{1}{2}\frac{\left(x-\mu_1\right)^2}{\sigma_1^{\,2}}\right)\right) = \log\left(\frac{1}{\sigma_2}\right) + \log\left(\exp\left(-\frac{1}{2}\frac{\left(x-\mu_2\right)^2}{\sigma_2^{\,2}}\right)\right)$$

$$\log\left(\frac{1}{\sigma_1}\right) - \frac{1}{2\sigma_1^{\,2}}\left(x-\mu_1\right)^2 = \log\left(\frac{1}{\sigma_2}\right) - \frac{1}{2\sigma_2^{\,2}}\left(x-\mu_2\right)^2$$

$$\log\left(\frac{1}{\sigma_1}\right) - \frac{1}{2\sigma_1^{\,2}}\left(x^2 - 2\mu_1 x + \mu_1^{\,2}\right) = \log\left(\frac{1}{\sigma_2}\right) - \frac{1}{2\sigma_2^{\,2}}\left(x^2 - 2\mu_2 x + \mu_2^{\,2}\right)$$

$$\left(-\frac{1}{2\sigma_1^{\,2}} + \frac{1}{2\sigma_2^{\,2}}\right)x^2 + \left(\frac{\mu_1}{\sigma_1^{\,2}} - \frac{\mu_2}{\sigma_2^{\,2}}\right)x + \left(\log\left(\frac{1}{\sigma_1}\right) - \frac{\mu_1^{\,2}}{2\sigma_1^{\,2}} - \log\left(\frac{1}{\sigma_2}\right) + \frac{\mu_2^{\,2}}{2\sigma_2^{\,2}}\right) = 0$$

$$a = -\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} = -\frac{1}{2*1} + \frac{1}{2*1,000,000} = -\frac{999,999}{2,000,000}$$

$$b = \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} = \frac{0}{1} - \frac{1}{1,000,000} = -\frac{1}{1,000,000}$$

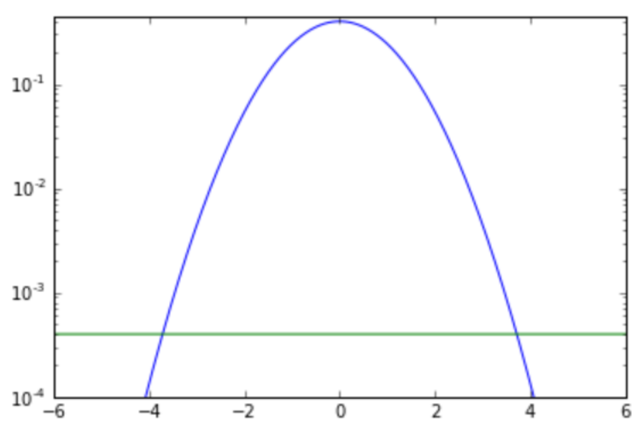
$$c = \log\left(\frac{1}{\sigma_1}\right) - \frac{\mu_1^2}{2\sigma_1^2} - \log\left(\frac{1}{\sigma_2}\right) + \frac{\mu_2^2}{2\sigma_2^2} = \log\left(\frac{1}{1}\right) - \frac{0}{2*1} - \log\left(\frac{1}{1,000}\right) + \frac{1}{2*1,000,000}$$

$$= -\log\left(\frac{1}{1,000}\right) + \frac{1}{2,000,000}$$

In [1]:

```
import numpy
from math import log, sqrt
a = numpy.float64(- 999999.0 / 2000000)
b = numpy.float64(- 1.0 / 1000000)
c = numpy.float64(- log(1.0 / 1000) + 1.0 / 2000000)
x = [ (- b + sign * sqrt(b**2 - 4 * a * c)) / (2 * a) for sign in [ 1, -1 ] ]
from scipy.stats import norm
pdf1 = norm.pdf(x, 0, 1)
pdf2 = norm.pdf(x, 1, 1000)
print(str(round(x[0], 9)) + "\t" + str(round(x[1], 9)))
print(str(round(pdf1[0], 9)) + "\t" + str(round(pdf1[1], 9)))
print(str(round(pdf2[0], 9)) + "\t" + str(round(pdf2[1], 9)))
from matplotlib import pyplot
%matplotlib inline
pyplot.yscale("log")
pyplot.ylim([ .0001, .45 ])
input = numpy.linspace(-6, 6, 1200)
pyplot.plot(input, norm.pdf(input, 0, 1))
pyplot.plot(input, norm.pdf(input, 1, 1000))
pyplot.show()
```

```
-3.716925182	3.716923182
0.000398938	0.000398941
0.000398938	0.000398941
```



So $R_1 = \{x : -3.716925 \le x \le 3.716923\}$.

b. Now suppose $\sigma_2 = 1$ (and all other parameters remain the same). What is R_1 in this case?

$$a = -\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} = -\frac{1}{2*1} + \frac{1}{2*1} = 0$$

$$b = \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} = \frac{0}{1} - \frac{1}{1} = -1$$

$$c = \log\left(\frac{1}{\sigma_1}\right) - \frac{\mu_1^2}{2\sigma_1^2} - \log\left(\frac{1}{\sigma_2}\right) + \frac{\mu_2^2}{2\sigma_2^2} = \log\left(\frac{1}{1}\right) - \frac{0}{2*1} - \log\left(\frac{1}{1}\right) + \frac{1}{2} = \frac{1}{2}$$

Since $a\,x^2 + b\,x + c = 0$ and $a = 0$, $x = -\frac{c}{b} = -\frac{1/2}{-1} = \frac{1}{2}$.

So $R_1 = \{x : x \le 0.5\}$.